EXAMPLE Addition of Power Series

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

as one power series.

SOLUTION In order to add the two series given in summation notation, it is necessary that both indices of summation start with the same number and that the powers of x in each series be "in phase," in other words, if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power. Note that in the given problem, the first series starts with x^0 whereas the second series starts with x^1. By writing the first term of the first series outside of the summation notation,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 2\cdot 1 c_2 x^0 + \sum_{n=3}^{\infty} \overset{\substack{\text{series starts}\\ \text{with } x\\ \text{for } n=3}}{n(n-1)c_n x^{n-2}} - \sum_{n=0}^{\infty} \overset{\substack{\text{series starts}\\ \text{with } x\\ \text{for } n=0}}{c_n x^{n+1}} \qquad (3)$$

we see that both series on the right side start with the same power of x, namely, x^1. Now to get the same summation index we are inspired by the exponents of x; we let $k = n - 2$ in the first series and at the same time let $k = n + 1$ in the second series. For $n = 3$ in $k = n - 2$ we get $k = 1$, and for $n = 0$ in $k = n + 1$ we get $k = 1$, and so the right-hand side of (3) becomes

same

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k. \qquad (4)$$

same

Remember the summation index is a "dummy" variable; the fact that $k = n - 2$ in one case and $k = n + 1$ in the other should cause no confusion if you keep in mind that it is the *value* of the summation index that is important. In both cases k takes on the same successive values $k = 1, 2, 3, \ldots$ when n takes on the values $n = 2, 3, 4, \ldots$ for $k = n - 1$ and $n = 0, 1, 2, \ldots$ for $k = n + 1$. We are now in a position to add the series in (4) term-by-term:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - c_{k-1}]x^k. \quad (5)$$