Chapter 14

Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. For example, the following are all rational functions.

$$f(x) = \frac{x}{3x+4} \quad g(x) = \frac{x^2+1}{3x-5} \quad h(x) = \frac{4x^5-4x^2-8}{x^3+x^2-x+1} \quad j(x) = \frac{x^6}{x^8+5x-\frac{1}{2}}$$

There is a huge variety of rational functions. In this course, we will concentrate our efforts exclusively toward understanding the simplest type of rational functions: *linear-to-linear rational functions*. Linear-to-linear rational functions are rational functions in which the numerator and the denominator are both linear polynomials. The following are linear-to-linear-rational functions.

$$k(x) = \frac{x}{3x+4} \quad m(x) = \frac{5x-6}{2x+1} \quad n(x) = \frac{0.34x-0.113}{x-1} \quad p(x) = \frac{4x+\frac{3}{4}}{\frac{5}{8}x-1.117}$$

The simplest example of a linear-to-linear rational function is $f(x) = \frac{1}{x}$ whose graph is shown in Figure 14.1. This is an important example for the study of this class of functions, as we shall see.

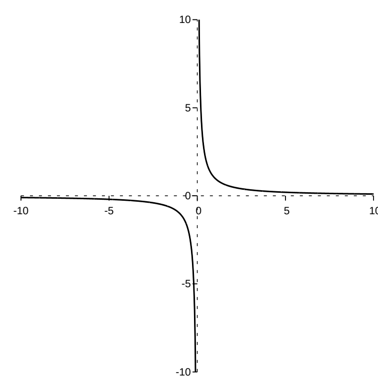


Figure 14.1: The graph of $f(x) = \frac{1}{x}$.

Let's consider the graph of this function, $f(x) = \frac{1}{x}$. We first begin by considering the domain of f. Since the numerator of $\frac{1}{x}$ is a constant, and the denominator is just x, the only way we can run into difficulty when evaluating this function is if we try to divide by zero; that is, the only value of x not in the domain of this function is zero. The function is defined for all x except $x = 0$. At $x = 0$, there must be a gap, or hole, in the graph.

To get the graph started we might simply give ourselves a point on the graph. For instance, we see that $f(1) = 1/1 = 1$, so the point $(1,1)$ is on the graph. Then, if we use larger values of x, we see that $1/x$ becomes smaller as x grows. For instance, $f(2) = 1/2$, $f(10) = 1/10$, and $f(1000) = 1/1000$. In addition, we see that, other than the fact that $1/x > 0$ for positive x, there is nothing preventing us from making $1/x$ as small as we want simply by taking x big enough. Want $1/x$ to be less than 0.001? Just pick x bigger than 1000. Want $1/x$ to be less than 0.000001? Just use x bigger than 1,000,000.

What this means graphically is that as x gets bigger (starting from $x = 1$), the curve $y = 1/x$ gets closer and closer to the x-axis. As a result, we say that the x-axis is a *horizontal asymptote* for this function.

We see the same behavior for negative values of x. If x is large, and negative (think -1000, or -1000000), then $1/x$ is very small (i.e., close to zero), and it gets smaller the larger x becomes. Graphically, this means that as x gets large in the negative direction, the curve $y = 1/x$ gets closer and closer to the x-axis. We say that, in both the positive and negative directions, $y = 1/x$ is *asymptotic* to the x-axis.

A similar thing happens when we consider x near zero. If x is a small positive number (think 1/2, or 1/10, or 1/10000), then $1/x$ is a large positive number. What's more, if we think of x as getting closer to zero, $1/x$ gets bigger and bigger. Plus, there is no bound on how big we can make $1/x$ simply by taking x as close to zero as we need to.

For instance, can $1/x$ be as big as 10000? All you need to do is pick a positive x smaller than 1/10000.

Graphically, what this means is that as x approaches zero from the positive side, the y value gets larger and larger. As a result, the curve approaches the y-axis as y gets larger. We say that the y-axis is a *vertical asymptote* for the curve $y = 1/x$.

We see the same phenomenon as x approaches zero from the negative side: $y = 1/x$ gets larger in the negative direction (i.e., it gets more and more negative). The curve gets closer and closer to the negative y-axis as y becomes more and more negative. Again, we say that the $y - axis$ is a *vertical asymptote* for the curve $y = 1/x$.

It turns out that every *linear-to-linear* rational function has a graph that looks essentially the same as the graph of $y = 1/x$. Let's see why.

Consider the linear-to-linear rational function $f(x) = \frac{ax+b}{cx+d}$. If we divide $cx + d$ into $ax + b$, the result is

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$

which we can rewrite as

$$f(x) = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \cdot \frac{1}{cx+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \cdot \frac{1}{c(x + \frac{d}{c})} = \frac{a}{c} + \left(\frac{bc - ad}{c^2}\right) \cdot \frac{1}{x + \frac{d}{c}}.$$

If we now let

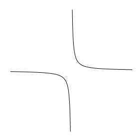
$$A = \frac{a}{c}\ ,\ B = \left(\frac{bc - ad}{c^2}\right),\ \text{and}\ C = \frac{d}{c}\ ,$$

then we can write

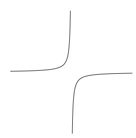
$$f(x) = A + B\frac{1}{x + C}.$$

If we let $g(x) = 1/x$ then we have shown that $f(x) = A + Bg(x + C)$, and so the graph of the function f is just a horizontally shifted, vertically shifted and vertically dilated version of the graph of g. Also, if B turned out to be negative the graph would be vertically flipped, too.

Why is that useful? It means that the graph of a linear-to-linear rational function can only take one of two forms. Either it looks like this:



Or like this:



We can sketch an accurate graph of a linear-to-linear rational function by sketching the asymptotes and then sketching in just one point on the graph. That will be enough information to nail down a decent sketch. We can always plot more points to give us more precision, but one point is enough to capture the essence of the graph.

Given a linear-to-linear function that we wish to graph, we must first determine the asymptotes. There will be one horizontal asymptote and one vertical asymptote.

The vertical asymptote will be a vertical line with equation $x = k$ where k is the one x value which is not in the domain of the function. That is, find the value of x which makes the denominator zero and that will tell you the vertical asymptote.

The horizontal asymptote is a little more involved. However, we can quickly get to a shortcut. The essence of a horizontal asymptote is that it describes what value the function is approximately equal to for very large values of x. To study what a linear-to-linear function is like when x is very large, we can perform the following algebraic manipulation:

$$f(x) = \frac{ax+b}{cx+d} = \frac{ax+b}{cx+d} \cdot \frac{1/x}{1/x} = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}$$

While dividing by x is troublesome if x equals zero, here we are assuming x is very large, so it is certainly *not* zero.

Now, consider this last expression in the above equation. If x is very large, then

$$\frac{b}{x} \approx 0$$

(where $\approx$ means "is approximately") and likewise

$$\frac{d}{x} \approx 0.$$

Hence, when x is very large,

$$f(x) \approx \frac{a+0}{c+0} = \frac{a}{c}.$$

We can interpret this by saying that when x is very large, the function $f(x)$ is is close to a constant, and that constant is $\frac{a}{c}$. Thus, the horizontal asymptote of $f(x) = \dfrac{ax+b}{cx+d}$ is the horizontal line $y = \frac{a}{c}$.

Example 14.0.1. *Sketch the graph of the function* $f(x) = \frac{3x-1}{2x+7}$.

Solution. We begin by finding the asymptotes of f.

The denominator is equal to zero when $2x + 7 = 0$, i.e., when $x = -7/2$. As a result, the vertical asymptote for this function is the vertical line $x = -7/2$.

By taking the ratio of the coefficients of x in the numerator and denominator, we can find that the horizontal asymptote is the horizontal line

$$y = \frac{3}{2}.$$

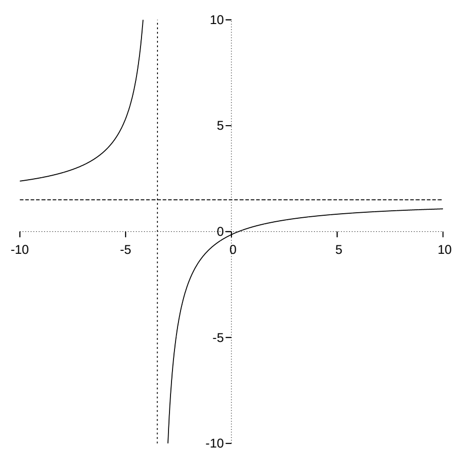


Figure 14.2: The graph of $f(x) = \frac{3x-1}{2x+7}$.

We then sketch these two asymptotes. The last thing we need is a single point. For instance, we may evaluate $f(0)$:

$$f(0) = \frac{-1}{7}$$

and so the point $(0, -1/7)$ is on the graph. With this information, we know that the curve lies *below* the horizontal asymptote to the right of the vertical asymptote, and consequently the curve lies *above* the horizontal asymptote to the left of the vertical asymptote.

We graph the result in Figure 14.2.

14.1 Modeling with Linear-to-linear Rational Functions

As we have done with other sorts of functions, such as linear and quadratic, we can also model using linear-to-linear rational functions. One reason for using this type of function is their asymptotic nature. Many changing quantities in the world continually increase or decrease, but with bounds on how large or small they can get. For instance, a population may steadily decrease, but a population can never be negative. Conversely, a population may steadily increase, but due to environmental and other factors we may hypothesize that the population will always stay below some upper bound. As a result, the population may "level off". This leveling off behavior is exemplified by the asymptotic nature of the linear-to-linear rational functions, and so this type of function provides a way to model such behavior.

Given any linear-to-linear rational function, we can always divide the numerator and the denominator by the coefficient of x in the denominator. In this way, we can always assume that the coefficient of x in the denominator of a function we seek is 1. This is illustrated in the next example.

Example 14.1.1. *Let* $f(x) = \frac{2x+3}{5x-7}$*. Then*

$$\begin{aligned} f(x) &= \frac{2x+3}{5x-7} \cdot \frac{\frac{1}{5}}{\frac{1}{5}} \\ &= \frac{\frac{2}{5}x + \frac{3}{5}}{x - \frac{7}{5}}. \end{aligned}$$

Thus, in general, when we seek a linear-to-linear rational function, we will be looking for a function of the form

$$f(x) = \frac{ax+b}{x+c}$$

and thus there are three parameters we need to determine.

Note that for a function of this form, the horizontal asymptote is $y = a$ and the vertical asymptote is $x = -c$.

Since these functions have three parameters (i.e., a, b and c), we will need three pieces of information to nail down the function.

There are essentially three types of modeling problems that require the determination of a linear-to-linear function. The three types are based on the kind of information given about the function. The three types are:

1. You know three points the the graph of the function passes through;
2. You know one of the function's asymptotes and two points the graph passes through;
3. You know both asymptotes and one point the graph passes through.

Notice that in all cases you know **three pieces of information**. Since a linear-to-linear function is determined by three parameters, this is exactly the amount of information needed to determine the function.

The worst case, in terms of the amount of algebra you need to do, is the first case. Let's look at an example of the algebra involved with this sort.

Example 14.1.2. *Find the linear-to-linear rational function* $f(x)$ *such that* $f(10) = 20$, $f(20) = 32$ *and* $f(25) = 36$.

Solution. Since $f(x)$ is a linear-to-linear rational function, we know

$$f(x) = \frac{ax + b}{x + c}$$

for constants a, b, and c. We need to find a, b and c.

We know three things.

First, $f(10) = 20$. So

$$f(10) = \frac{10a + b}{10 + c} = 20,$$

which we can rewrite as

$$10a + b = 200 + 20c. \tag{14.1}$$

Second, $f(20) = 32$. So

$$f(20) = \frac{20a + b}{20 + c} = 32,$$

which we can rewrite as

$$20a + b = 640 + 32c. \tag{14.2}$$

Third, $f(25) = 36$. So

$$f(25) = \frac{25a + b}{25 + c} = 36,$$

which we can rewrite as

$$25a + b = 900 + 36c. \tag{14.3}$$

These three numbered equations are enough algebraic material to solve for a, b, and c. Here is one way to do that.

Subtract equation 14.1 from equation 14.2 to get

$$10a = 440 + 12c \tag{14.4}$$

and subtract equation 14.2 from equation 14.3 to get

$$5a = 260 + 4c \tag{14.5}$$

Note that we've eliminated b. Now multiply this last equation by 2 to get

$$10a = 520 + 8c$$

Subtract equation 14.4 from this to get

$$0 = 80 - 4c$$

which easily give us $c = 20$.

Plugging this value into equation 14.4, we can find $a = 68$, and then we can find $b = -80$.

Thus,

$$f(x) = \frac{68x - 80}{x + 20}.$$

We can check that we have done the algebra correctly by evaluating $f(x)$ at $x = 10$, $x = 20$ and $x = 25$. If we get $f(10) = 20$, $f(20) = 32$ and $f(25) = 36$, then we'll know our work is correct.

□

Algebraically, this was the worst situation of the three, since it required the most algebra. If, instead of knowing three points, we know one or both of the asymptotes, then we can easily find a and/or c, and so cut down on the amount of algebra needed. However, the method is essentially identical.

Let's now apply these ideas to a real world problem.

Example 14.1.3. *Clyde makes extra money selling tickets in front of the Safeco Field. The amount he charges for a ticket depends on how many he has. If he only has one ticket, he charges \$100 for it. If he has 10 tickets, he charges \$80 a piece. But if he has a large number of tickets, he will sell them for \$50 each. How much will he charge for a ticket if he holds 20 tickets?*

Solution. We want to give a linear-to-linear rational function relating the price of a ticket y to the number of tickets x that Clyde is holding. As we saw above, we can assume the function is of the form

$$y = \frac{ax + b}{x + c}$$

where a, b and c are numbers. Note that $y = a$ is the horizontal asymptote. When x is very large, y is close to 50. This means the line $y = 50$ is a horizontal asymptote. Thus $a = 50$ and

$$y = \frac{50x + b}{x + c}.$$

Next we plug in the point $(1,100)$ to get a linear equation in b and c.

$$\begin{aligned} 100 &= \frac{50 \cdot 1 + b}{1 + c} \\ 100 \cdot (1 + c) &= 50 + b \\ 50 &= b - 100c \end{aligned}$$

Similarly, plugging in $(10,80)$ and doing a little algebra (do it now!) gives another linear equation $300 = b - 80c$. Solving these two linear equations simultaneously gives $c = 12.5$ and $b = 1300$. Thus our function is

$$y = \frac{50x + 1300}{x + 12.5}$$

and, if Clyde holds 20 tickets, he will charge

$$y = \frac{50 \cdot 20 + 1300}{20 + 12.5} = \$70.77$$

per ticket. □

14.2 Summary

- Every linear-to-linear rational function has a graph which is a shifted, scaled version of the curve $y = 1/x$. As a result, they have one vertical asymptote, and one horizontal asymptote.

- Every linear-to-linear rational function f can be expressed in the form

  $$f(x) = \frac{ax + b}{x + c}.$$

  This function has horizontal asymptote $y = a$ and vertical asymptote $x = -c$.

14.3 Exercises

Problem 14.1. Give the domain of each of the following functions. Find the x- and y-intercepts of each function. Sketch a graph and indicate any vertical or horizontal asymptotes. Give equations for the asymptotes.

(a) $f(x) = \frac{2x}{x-1}$ (b) $g(x) = \frac{3x+2}{2x-5}$

(c) $h(x) = \frac{x+1}{x-2}$ (d) $j(x) = \frac{4x-12}{x+8}$

(e) $k(x) = \frac{8x+16}{5x-\frac{1}{2}}$ (f) $m(x) = \frac{9x+24}{35x-100}$

Problem 14.2. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4.

(a) Give a linear-to-linear rational model relating the meter reading y to how many feet x Oscar has gone into the room.

(b) How far must he go for the meter to reach 10? 100?

(c) Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

Problem 14.3. In 1975 I bought an old Martin ukulele for \$200. In 1995 a similar uke was selling for \$900. In 1980 I bought a new Kamaka uke for \$100. In 1990 I sold it for \$400.

(a) Give a linear model relating the price p of the Martin uke to the year t. Take $t = 0$ in 1975.

(b) Give a linear model relating the price q of the Kamaka uke to the year t. Again take $t = 0$ in 1975.

(c) When is the value of the Martin twice the value of the Kamaka?

(d) Give a function $f(t)$ which gives the ratio of the price of the Martin to the price of the Kamaka.

(e) In the long run, what will be the ratio of the prices of the ukuleles?

Problem 14.4. Isobel is producing and selling casette tapes of her rock band. When she had sold 10 tapes, her net profit was \$6. When she had sold 20 tapes, however, her net profit had shrunk to \$4 due to increased production expenses. But when she had sold 30 tapes, her net profit had rebounded to \$8.

(a) Give a quadratic model relating Isobel's net profit y to the number of tapes sold x.

(b) Divide the profit function in part (a) by the number of tapes sold x to get a model relating average profit w per tape to the number of tapes sold.

(c) How many tapes must she sell in order to make \$1.20 per tape in net profit?

Problem 14.5. Find the linear-to-linear function whose graph passes through the points $(1,1)$, $(5,2)$ and $(20,3)$. What is its horizontal asymptote?

Problem 14.6. Find the linear-to-linear function whose graph has $y = 6$ as a horizontal asymptote and passes through $(0,10)$ and $(3,7)$.

Problem 14.7. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score is a linear-to-linear function of the number of hours that you study.

If you want a score of 80%, how long do you need to study?

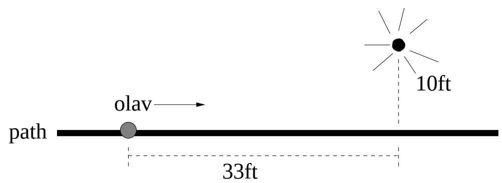
Problem 14.8. A street light is 10 feet above a straight bike path. Olav is bicycling down the path at a rate of 15 MPH. At midnight, Olav is 33 feet from the point on the bike path directly below the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source.

(a) From 20 feet away, the street light has an intensity of 1 candle. What is k?

(b) Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.

(c) When will the light on Olav have maximum intensity?

(d) When will the intensity of the light be 2 candles?



Problem 14.9. For each of the following find the linear to linear function $f(x)$ satisfying the given requirements:

(a) $f(0) = 0, f(10) = 10, f(20) = 15$

(b) $f(0) = 10, f(5) = 4, f(20) = 3$

(c) $f(10) = 20, f(30) = 25$, and the graph of $f(x)$ has $y = 30$ as its horizontal asymptote

Problem 14.10. The number of customers in a local dive shop depends on the amount of money spent on advertising. If the shop spends nothing on advertising, there will be 100 customers/day. If the shop spends \$100, there will be 200 customers/day. As the amount spent on advertising increases, the number of customers/day increases and approaches (but never exceeds) 400 customers/day.

(a) Find a linear to linear rational function $y = f(x)$ that calculates the number y of customers/day if \$$x$ is spent on advertising.

(b) How much must the shop spend on advertising to have 300 customers/day.

(c) Sketch the graph of the function $y = f(x)$ on the domain $0 \le x \le 5000$.

(d) Find the rule, domain and range for the inverse function from part (c). Explain in words what the inverse function calculates.