

1.1 SETS OF REAL NUMBERS AND THE CARTESIAN COORDINATE PLANE

1.1.1 SETS OF NUMBERS

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience¹ has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of ‘set theory’ and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

Definition 1.1. A **set** is a well-defined collection of objects which are called the ‘elements’ of the set. Here, ‘well-defined’ means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word “smolko” is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set.² In general, there are three ways to describe sets. They are

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define a set.
2. **The Roster Method:** Begin with a left brace ‘{’, list each element of the set *only once* and then end with a right brace ‘}’.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a “dummy variable” such as x .

For example, let S be the set described *verbally* as the set of letters that make up the word “smolko”. A **roster** description of S would be $\{s, m, o, l, k\}$. Note that we listed ‘o’ only once, even though it

¹...to be read as ‘good, solid feedback from colleagues’ ...

²For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous [Russell’s Paradox](#).

appears twice in “smolko.” Also, the *order* of the elements doesn’t matter, so $\{k, l, m, o, s\}$ is also a roster description of S . A **set-builder** description of S is:

$$\{x \mid x \text{ is a letter in the word “smolko”}.\}$$

The way to read this is: ‘The set of elements x such that x is a letter in the word “smolko.”’ In each of the above cases, we may use the familiar equals sign ‘=’ and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word “smolko”}.\}$. Clearly m is in S and q is not in S . We express these sentiments mathematically by writing $m \in S$ and $q \notin S$. Throughout your mathematical upbringing, you have encountered several famous sets of numbers. They are listed below.

Sets of Numbers

1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements. Like the number ‘0,’ it plays a vital role in mathematics.^a
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \dots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, ‘and so forth’.
3. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \dots\}$
4. The **Integers**: $\mathbb{Z} = \{\dots, -1, -2, -1, 0, 1, 2, 3, \dots\}$
5. The **Rational Numbers**: $\mathbb{Q} = \{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\}$. Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers^b is:

$$\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation}.\}$$

6. The **Real Numbers**: $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation}.\}$
7. The **Irrational Numbers**: $\mathbb{P} = \{x \mid x \text{ is a non-rational real number}.\}$ Said another way, an irrational number is a decimal which neither repeats nor terminates.^c
8. The **Complex Numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ Despite their importance, the complex numbers play only a minor role in the text.^d

^a... which, sadly, we will not explore in this text.

^bSee Section 9.2.

^cThe classic example is the number π (See Section 10.1), but numbers like $\sqrt{2}$ and $0.101001000100001\dots$ are other fine representatives.

^dThey first appear in Section 3.4 and return in Section 11.7.

It is important to note that every natural number is a whole number, which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for \mathbb{Q}) and the rational numbers are all real numbers, since they possess decimal representations.³ If we take $b = 0$ in the

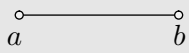
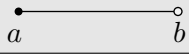
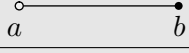
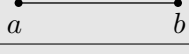
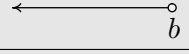

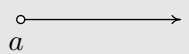
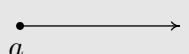
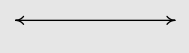
³Long division, anyone?

above definition of \mathbb{C} , we see that every real number is a complex number. In this sense, the sets \mathbb{N} , \mathbb{W} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} are ‘nested’ like [Matryoshka dolls](#).



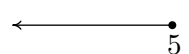
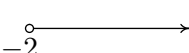
For the most part, this textbook focuses on sets whose elements come from the real numbers \mathbb{R} . Recall that we may visualize \mathbb{R} as a line. Segments of this line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, ‘[’ or ‘]’, if the endpoint is included in the interval and use a filled-in or ‘closed’ dot to indicate membership in the interval. Otherwise, we use parentheses, ‘(’ or ‘)’ and an ‘open’ circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

Interval Notation

Let a and b be real numbers with $a < b$.

| Set of Real Numbers | Interval Notation | Region on the Real Number Line |
|------------------------------|---------------------|--|
| $\{x \mid a < x < b\}$ | (a, b) |  |
| $\{x \mid a \leq x < b\}$ | $[a, b)$ |  |
| $\{x \mid a < x \leq b\}$ | $(a, b]$ |  |
| $\{x \mid a \leq x \leq b\}$ | $[a, b]$ |  |
| $\{x \mid x < b\}$ | $(-\infty, b)$ |  |
| $\{x \mid x \leq b\}$ | $(-\infty, b]$ |  |
| $\{x \mid x > a\}$ | (a, ∞) |  |
| $\{x \mid x \geq a\}$ | $[a, \infty)$ |  |
| \mathbb{R} | $(-\infty, \infty)$ |  |

For an example, consider the sets of real numbers described below.

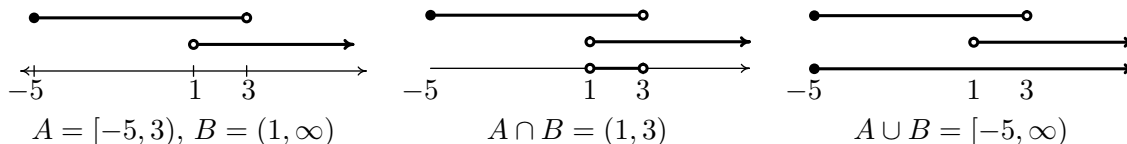
| Set of Real Numbers | Interval Notation | Region on the Real Number Line |
|-------------------------------|-------------------|--|
| $\{x \mid 1 \leq x < 3\}$ | $[1, 3)$ |  |
| $\{x \mid -1 \leq x \leq 4\}$ | $[-1, 4]$ |  |
| $\{x \mid x \leq 5\}$ | $(-\infty, 5]$ |  |
| $\{x \mid x > -2\}$ | $(-2, \infty)$ |  |

We will often have occasion to combine sets. There are two basic ways to combine sets: **intersection** and **union**. We define both of these concepts below.

Definition 1.2. Suppose A and B are two sets.

- The **intersection** of A and B : $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- The **union** of A and B : $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ (or both)}\}$

Said differently, the intersection of two sets is the overlap of the two sets – the elements which the sets have in common. The union of two sets consists of the totality of the elements in each of the sets, collected together.⁴ For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1, 2, 3, 4, 6\}$. If $A = [-5, 3)$ and $B = (1, \infty)$, then we can find $A \cap B$ and $A \cup B$ graphically. To find $A \cap B$, we shade the overlap of the two and obtain $A \cap B = (1, 3)$. To find $A \cup B$, we shade each of A and B and describe the resulting shaded region to find $A \cup B = [-5, \infty)$.



While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

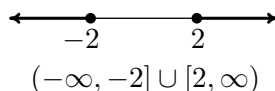
⁴The reader is encouraged to research [Venn Diagrams](#) for a nice geometric interpretation of these concepts.

Example 1.1.1. Express the following sets of numbers using interval notation.

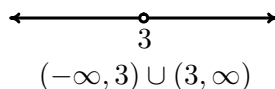
1. $\{x \mid x \leq -2 \text{ or } x \geq 2\}$
2. $\{x \mid x \neq 3\}$
3. $\{x \mid x \neq \pm 3\}$
4. $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

Solution.

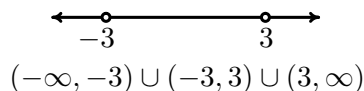
1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. Since we are looking to describe the real numbers x in one of these *or* the other, we have $\{x \mid x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.



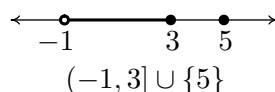
2. For the set $\{x \mid x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in either one of these intervals *or* the other, we have that $\{x \mid x \neq 3\} = (-\infty, 3) \cup (3, \infty)$



3. For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.



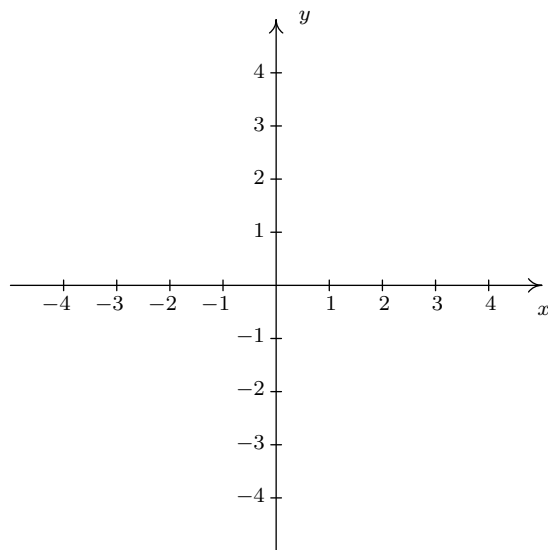
4. Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$, we get one interval, $(-1, 3]$ along with a single number, or point, $\{5\}$. While we *could* express the latter as $[5, 5]$ (Can you see why?), we choose to write our answer as $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.



□

1.1.2 THE CARTESIAN COORDINATE PLANE

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.⁵ Imagine two real number lines crossing at a right angle at 0 as drawn below.



The horizontal number line is usually called the **x -axis** while the vertical number line is usually called the **y -axis**.⁶ As with the usual number line, we imagine these axes extending off indefinitely in both directions.⁷ Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x -axis to P and extending a horizontal line from the y -axis to P . This process is sometimes called ‘projecting’ the point P to the x - (respectively y -) axis. We then describe the point P using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or **x -coordinate** and the second is called the **ordinate** or **y -coordinate**.⁸ Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates**⁹ of the point P . In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of ‘the point $(2, -4)$.’ We can think of $(2, -4)$ as instructions on how to

⁵So named in honor of [René Descartes](#).

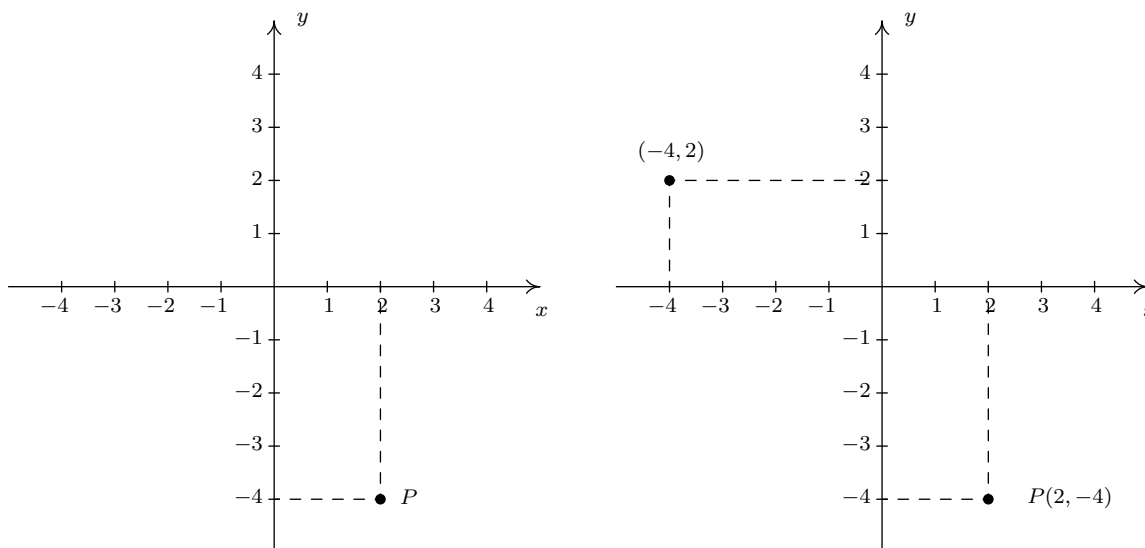
⁶The labels can vary depending on the context of application.

⁷Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y .

⁸Again, the names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the t -axis. The first number in the ordered pair would then be the t -coordinate.

⁹Also called the ‘rectangular coordinates’ of P – see Section 11.4 for more details.

reach P from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important – if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.



When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

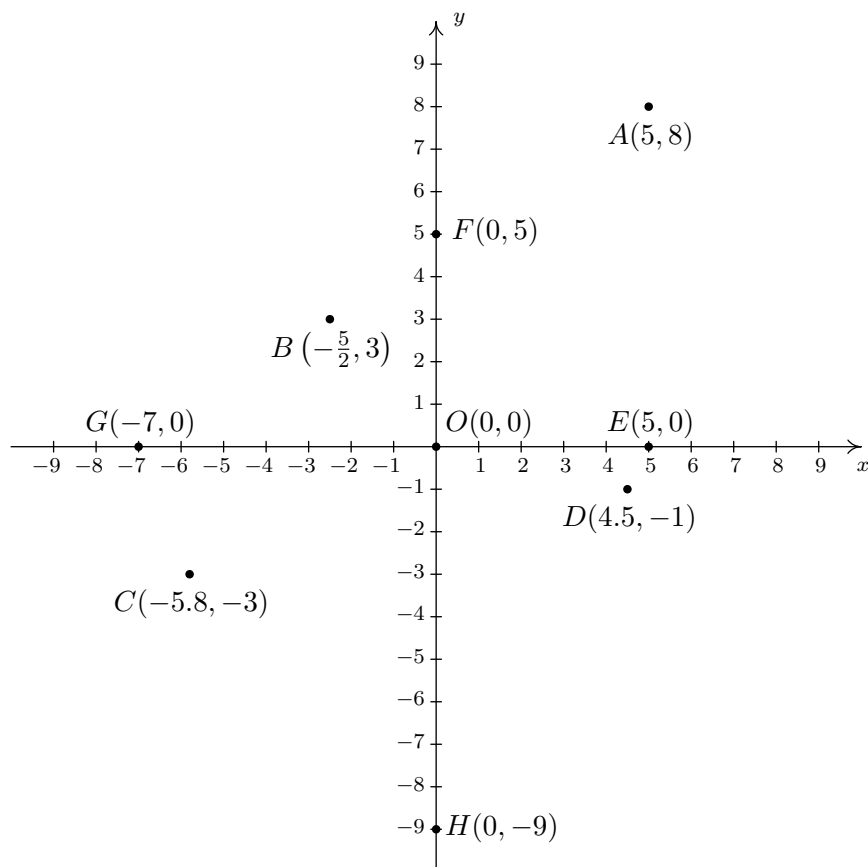
Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x -axis if and only if $y = 0$.
- (x, y) lies on the y -axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

Example 1.1.2. Plot the following points: $A(5, 8)$, $B(-\frac{5}{2}, 3)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.¹⁰

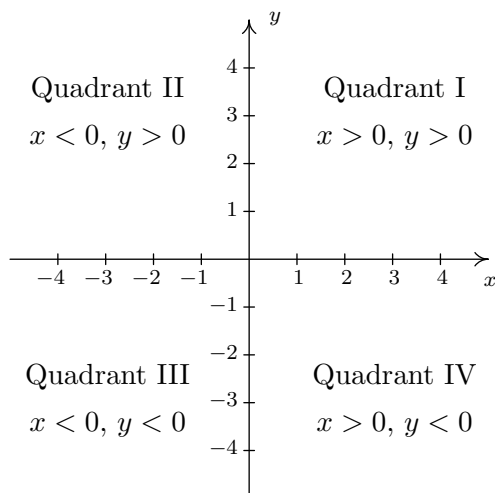
Solution. To plot these points, we start at the origin and move to the right if the x -coordinate is positive; to the left if it is negative. Next, we move up if the y -coordinate is positive or down if it is negative. If the x -coordinate is 0, we start at the origin and move along the y -axis only. If the y -coordinate is 0 we move along the x -axis only.

¹⁰The letter O is almost always reserved for the origin.



□

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



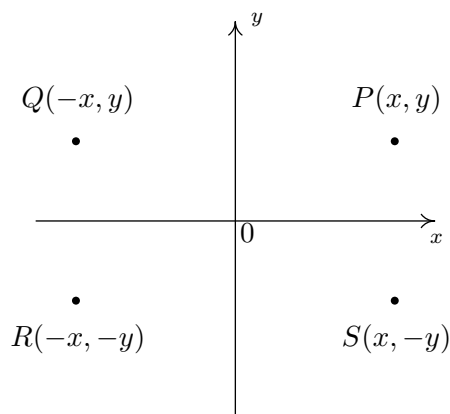
For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x -axis (if $y = 0$) or on the positive or negative y -axis (if $x = 0$). For example, $(0, 4)$ lies on the positive y -axis whereas $(-117, 0)$ lies on the negative x -axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**.¹¹ There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.3. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the x -axis** if $a = c$ and $b = -d$
- **symmetric about the y -axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x -axis, as are Q and R ; P and Q are symmetric about the y -axis, as are R and S ; and P and R are symmetric about the origin, as are Q and S .

Example 1.1.3. Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

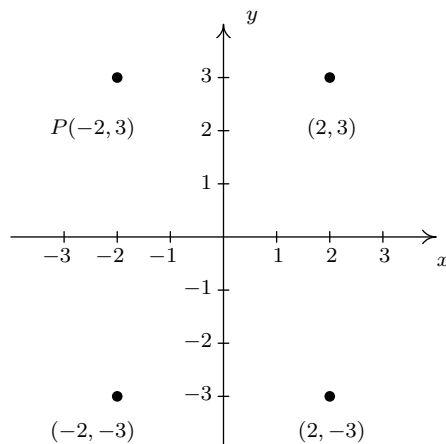
1. x -axis
2. y -axis
3. origin

Check your answer by plotting the points.

Solution. The figure after Definition 1.3 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x - and/or y -coordinates of $P(-2, 3)$.

¹¹According to Carl. Jeff thinks symmetry is overrated.

1. To find the point symmetric about the x -axis, we replace the y -coordinate with its opposite to get $(-2, -3)$.
2. To find the point symmetric about the y -axis, we replace the x -coordinate with its opposite to get $(2, 3)$.
3. To find the point symmetric about the origin, we replace the x - and y -coordinates with their opposites to get $(2, -3)$.



One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2, 3)$ and pretend that the x -axis is a mirror, then the reflection of $(-2, 3)$ across the x -axis would lie at $(-2, -3)$. If we pretend that the y -axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the x -axis and then the y -axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

Reflections

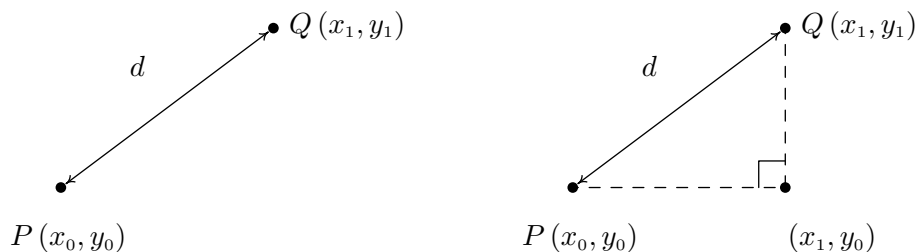
To reflect a point (x, y) about the:

- x -axis, replace y with $-y$.
- y -axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.1.3 DISTANCE IN THE PLANE

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, in the plane. By the **distance** d between P and Q , we mean the length of the line segment joining P with Q . (Remember, given any two distinct points in the plane, there is a unique line containing both

points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.



With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the [Pythagorean Theorem](#) gives us

$$\begin{aligned} |x_1 - x_0|^2 + |y_1 - y_0|^2 &= d^2 \\ (x_1 - x_0)^2 + (y_1 - y_0)^2 &= d^2 \end{aligned}$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation 1.1. The Distance Formula: The distance d between the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in [Exercise 35](#) to verify [Equation 1.1](#) for these cases.

Example 1.1.4. Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

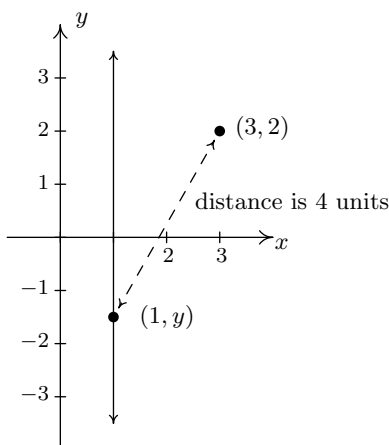
Solution.

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$. □

Example 1.1.5. Find all of the points with x -coordinate 1 which are 4 units from the point $(3, 2)$.

Solution. We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$. Visually,

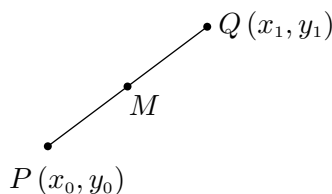


We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 1.1, yields

$$\begin{aligned}
 d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
 4 &= \sqrt{(1 - 3)^2 + (y - 2)^2} \\
 4 &= \sqrt{4 + (y - 2)^2} \\
 4^2 &= \left(\sqrt{4 + (y - 2)^2}\right)^2 && \text{squaring both sides} \\
 16 &= 4 + (y - 2)^2 \\
 12 &= (y - 2)^2 \\
 (y - 2)^2 &= 12 \\
 y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
 y - 2 &= \pm 2\sqrt{3} \\
 y &= 2 \pm 2\sqrt{3}
 \end{aligned}$$

We obtain two answers: $(1, 2 + 2\sqrt{3})$ and $(1, 2 - 2\sqrt{3})$. The reader is encouraged to think about why there are two answers. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q .



If we think of reaching M by going ‘halfway over’ and ‘halfway up’ we get the following formula.

Equation 1.2. The Midpoint Formula: The midpoint M of the line segment connecting $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right)$$

If we let d denote the distance between P and Q , we leave it as Exercise 36 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q . This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.6. Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution.

$$\begin{aligned} M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right) \\ &= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2} \right) = \left(-\frac{1}{2}, \frac{0}{2} \right) \\ &= \left(-\frac{1}{2}, 0 \right) \end{aligned}$$

The midpoint is $(-\frac{1}{2}, 0)$. □

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 2.1.

Example 1.1.7. If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a) .

Solution. To prove the claim, we use Equation 1.2 to find the midpoint

$$\begin{aligned} M &= \left(\frac{a + b}{2}, \frac{b + a}{2} \right) \\ &= \left(\frac{a + b}{2}, \frac{a + b}{2} \right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required. □