Appendix C: Uncertainty Formulae

For a function F of multiple variables x, each with uncertainty σ_x, the square of the uncertainty in F is given as

$$\sigma_F^2 = \Sigma_i \left(\frac{\delta F}{\delta x_i}\sigma_{x_i}\right)^2 \tag{1}$$

Addition rule: $F(x_1, x_2) = x_1 + x_2$

$$\begin{aligned} \sigma_F^2 &= \left(\frac{\delta F}{\delta x_1}\sigma_{x_1}\right)^2 + \left(\frac{\delta F}{\delta x_2}\sigma_{x_2}\right)^2 \\ &= \sigma_{x_1}^2 + \sigma_{x_2}^2 \end{aligned} \tag{2}$$

Example: Assume you must determine a total length L consisting of two measurements x_1 and x_2.

$$L = x_1 + x_2 \tag{3}$$

$$\sigma_L^2 = \left(\frac{\delta F}{\delta_{x_1}}\sigma_{x_1}\right)^2 + \left(\frac{\delta F}{\delta_{x_2}}\sigma_{x_2}\right)^2 \tag{4}$$

If $x_1 = (1.23 \pm 0.02)$ m and $x_2 = (4.17 \pm 0.01)$ m, then

$$\begin{aligned} \sigma_L^2 &= ((0.02)^2 + (0.01)^2)\ \mathrm{m}^2 \\ &= (4 \times 10^{-4} + 1 \times 10^{-4})\ \mathrm{m}^2 \\ &= 5 \times 10^{-4}\ \mathrm{m}^2 \\ \sigma_L &= 2.2 \times 10^{-2}\ \mathrm{m}, \text{ and} \\ L &= (5.40 \pm 0.02)\ \mathrm{m} \end{aligned} \tag{5}$$

Product rule: $\boldsymbol{F(\mathrm{A},\mathrm{B},\mathrm{C}) = k\mathrm{A}^{\mathrm{m}}\mathrm{B}^{\mathrm{n}}\mathrm{C}^{\mathrm{p}}}$

$$
\begin{aligned}
\sigma_F^2 &= \left(\frac{\delta F}{\delta A}\sigma_{\mathrm{A}}\right)^2 + \left(\frac{\delta F}{\delta B}\sigma_{\mathrm{B}}\right)^2 + \left(\frac{\delta F}{\delta C}\sigma_{\mathrm{C}}\right)^2 \\
&= \left(\frac{mF}{A}\sigma_{\mathrm{A}}\right)^2 + \left(\frac{nF}{B}\sigma_{\mathrm{B}}\right)^2 + \left(\frac{pF}{C}\sigma_{\mathrm{C}}\right)^2 \\
&= F^2\left[\left(\frac{m\sigma_A}{A}\right)^2 + \left(\frac{n\sigma_B}{B}\right)^2 + \left(\frac{p\sigma_C}{C}\right)^2\right] \\
\sigma_F &= F\sqrt{\left(\frac{m\sigma_A}{A}\right)^2 + \left(\frac{n\sigma_B}{B}\right)^2 + \left(\frac{p\sigma_C}{C}\right)^2}
\end{aligned}
\tag{6}
$$

Example: $F = k\frac{Q_1 Q_2}{r^2}$

$$
\sigma_F = F\sqrt{\left(\frac{\sigma_{Q1}}{Q1}\right)^2 + \left(\frac{\sigma_{Q2}}{Q_2}\right)^2 + \left(\frac{-2\sigma_r}{r}\right)^2} \tag{7}
$$

Assume $Q_1 = (6.1 \pm 0.4)\ \mu$C, $Q_2 = (4.7 \pm 0.3)\ \mu$C, $r = (0.025 \pm 0.003)$ m.

$F = 8.99 \times 10^9 \frac{(6.1\times10^{-6})(4.7\times10^{-6})}{0.025^2} = 4.1 \times 10^2$ N

$$
\begin{aligned}
\sigma_F &= 4.1 \times 10^2\sqrt{\left(\frac{0.4}{6.1}\right)^2 + \left(\frac{0.3}{4.7}\right)^2 + \left(\frac{(-2)(0.003)}{0.025}\right)^2} \\
&= 4.1 \times 10^2\sqrt{4.30 \times 10^{-3} + 4.07 \times 10^{-3} + 5.76 \times 10^{-2}} \\
&= (4.1 \times 10^2)(0.26) \\
&= 1.07 \times 10^2 \text{ N}
\end{aligned}
\tag{8}
$$

$F = (410 \pm 100)$ N