Answer to Essential Question 12.6: Consider Equation 12.8, $\omega = \sqrt{k/m}$. We can double the angular frequency by increasing the spring constant by a factor of 4, or by decreasing the mass by a factor of 4. Because the object is released from rest, the initial energy is all elastic potential energy, given by $U_i = 0.5\,k\,A^2$. We have not changed A, so if the total energy stayed the same we must not have changed the spring constant k. Thus we must have changed the mass.

12-7 The Simple Pendulum

Another classic simple harmonic motion system is the simple pendulum, which is an object with mass that swings back and forth on a string of negligible mass.

EXAMPLE 12.7 – Pendulum speed limit
A ball of mass m is fastened to a string with a length L. Initially the ball hangs vertically down from the string in its equilibrium position. The ball is then displaced so the string makes an angle of θ with the vertical, and then released from rest.
(a) What is the height of the ball above the equilibrium position when it is released?
(b) What is the speed of the ball when it passes through equilibrium?
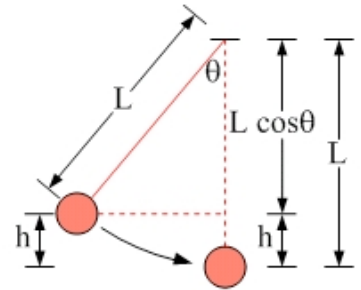

Figure 12.17: A diagram showing the point from which the ball is released and the equilibrium position.

SOLUTION
(a) Consider the geometry of the situation, shown in Figure 12.17. The key to finding the height of the ball is to consider the right-angled triangle in Figure 12.17. The vertical side of the triangle measures $L\cos\theta$. Because the string measures L, the height of the ball above equilibrium is $h = L - L\cos\theta = L(1 - \cos\theta)$.

(b) Let's apply energy conservation, starting as usual with: $K_i + U_i + W_{nc} = K_f + U_f$.
The initial point is the release point, while the final point is the equilibrium position.
   $K_i = 0$, because the ball is released from rest from the initial point.

   $W_{nc} = 0$, because there is no work being done by non-conservative forces.

   We can define the ball's gravitational potential energy to be zero at the equilibrium point, giving $U_f = 0$.

The equation thus reduces to $U_i = K_f$, which gives: $mgh = \dfrac{1}{2}mv_f^2$.

The mass cancels out, so the speed does not depend on the mass. Solving for the speed:
$v_f = \sqrt{2gh} = \sqrt{2gL(1 - \cos\theta)}$.

Note that we have seen this $v_f = \sqrt{2gh}$ result before, such as in cases in which an object falls straight down from rest, or when water leaks out a hole in a container.

Related End-of-Chapter Exercises: 11, 29.

EXPLORATION 12.7 – Torques on the pendulum
A simple pendulum consists of a ball of mass m that hangs down vertically from a string. The ball is displaced by an angle θ from equilibrium and released from rest.

Figure 12.18: Figure (a) shows the free-body diagram, with a force of tension and a force of gravity acting on the ball. Figure (b) shows the force of gravity in components, using a coordinate system in which one axis is parallel to the string.

Step 1 – *Draw a free-body diagram for the ball immediately after it is released.* The free-body diagram is drawn in Figure 12.18). There is a downward force of gravity, and a force of tension directed away from the ball along the string. Using a coordinate system aligned with the string, we can split the force of gravity into components, one component opposite to the tension and the other component giving an acceleration toward the equilibrium position (see Figure 12.18(b)).
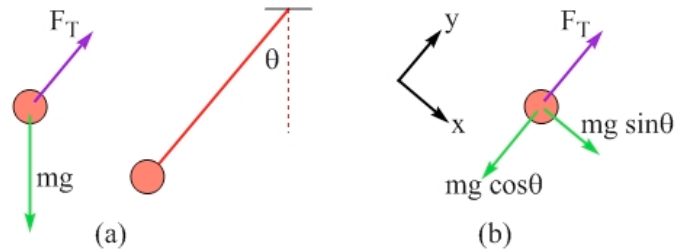
Step 2 – *Apply Newton's Second Law for Rotation to find a relationship between the angular acceleration of the ball and the ball's angular displacement (measured from the vertical).* Take torques about the axis perpendicular to the page passing through the upper end of the string. There is no torque about this axis from the tension or from the component of the force of gravity parallel to the string. The only torque comes from the component of the force of gravity that acts perpendicular to the string. Applying $\tau = r\,F\sin\phi$, where $r = L$, $F = mg\sin\theta$, and $\phi = 90°$, the torque has a magnitude of

$\tau = Lmg\sin\theta$. Taking counterclockwise to be positive, the torque is negative. Applying Newton's

Second Law for Rotation, $\sum \vec{\tau} = I\vec{\alpha}$, we get: $-Lmg\sin\theta = I\vec{\alpha}$.

The rotational inertia of the ball is $I = mL^2$, giving: $-Lmg\sin\theta = mL^2\vec{\alpha}$.

Canceling the mass (thus, the mass does not matter) and a factor of L gives:

$\vec{\alpha} = -\dfrac{g}{L}\sin\theta$.   (Equation 12.9: **Angular acceleration of a simple pendulum**)

Step 3 – *Use the small-angle approximation,* $\sin\theta \approx \theta$, ***to find an expression for the angular frequency of the pendulum.*** We can say that $\sin\theta \approx \theta$ if θ is given in radians, and the angle is less than about 10˚(about 1/6 radians). Using the small-angle approximation in Equation 12.9:

$\vec{\alpha} = -\dfrac{g}{L}\vec{\theta}$.   (Equation 12.10: **For a simple pendulum at small angles**)

The general simple harmonic relationship, $\vec{a} = -\omega^2\,\vec{x}$, can be transformed to an analogous general equation for rotational motion, $\vec{\alpha} = -\omega^2\vec{\theta}$. Equation 12.10 fits this form, so:

$\omega = \sqrt{\dfrac{g}{L}}$.   (Eq. 12.11: **Angular frequency for a simple pendulum at small angles**)

For a pendulum, gravity provides the restoring force, so it makes sense that the angular frequency is larger if g is larger. Conversely, increasing L means the pendulum has farther to travel to reach equilibrium, reducing the angular frequency.

Key ideas: For small-angle oscillations, the motion of a simple pendulum is simple harmonic. Large-angle oscillations are not simple harmonic because the restoring torque is not proportional to the angular displacement.   **Related End-of-Chapter Exercises: 57, 58.**

Essential Question 12.7: Compare the free-body diagram of a ball of mass *m,* hanging at rest from a string of length *L*, to that of the same system oscillating as a pendulum, when the ball passes through equilibrium. Make note of any differences between the two free-body diagrams.