Answer to Essential Question 11.7: Because Sarah and the merry-go-round stick together and move as one after the collision, the collision is completely inelastic.

11-8 Racing Shapes

Let's make use of the expression for rotational kinetic energy we derived in section 11-7, and apply it to analyze the motion of an object that rolls without slipping down a slope. The analysis can be done in terms of energy conservation (as we will do), or in terms of thinking about forces and torques and applying Newton's second law and Newton's second law for rotation. The analysis in those terms can be found on the accompanying web site.

EXPLORATION 11.8 – Racing shapes

You have various shapes, including a few different solid spheres, a few rings, and a few uniform disks and cylinders. The objects have various masses and radii. When you race the objects by releasing them from rest two at a time, they roll without slipping down an incline of constant angle. Our goal is to determine which object reaches the bottom of the incline in the shortest time. Let's analyze this for a generic object of mass *M*, radius *R*, and rotational inertia, about an axis through the center of mass, of cMR^2 .

Step 1 – ***Sketch a free-body diagram for the object as it rolls without slipping down the ramp.*** A diagram and a free-body diagram is shown in Figure 11.19. The Earth applies a downward force of gravity to the object, while the incline applies a contact force. We split the contact force into two forces, a normal force perpendicular to the incline and a force of friction directed up the slope. This is a static force of friction, because the object does not slip as it rolls. The force of static friction is directed up the slope, not because the motion of the object is down the slope, but because the object has a clockwise angular acceleration (its angular velocity is clockwise and increasing as it rolls down). Taking an axis through the center of the object, the static force of friction is the only force that can provide the torque associated with this angular acceleration – the other two forces pass through the center of the object and thus give no torque about that axis.

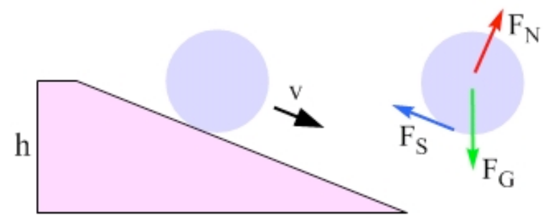

Figure 11.19: The diagram and free-body diagram of an object as it rolls without slipping down a ramp. A force of friction directed up the ramp provides the clockwise torque associated with the object's clockwise angular acceleration. The force of friction is static because the object does not slip as it rolls.

Step 2 – ***Let's analyze this in terms of energy conservation, using the same conservation of energy equation we used in previous chapters. Start by eliminating the terms that are zero in the equation.*** Recall that the energy conservation equation is: $K_i + U_i + W_{nc} = K_f + U_f$. The object is released from rest, so the initial kinetic energy K_i is zero. We can also define the bottom of the incline to be the zero level for gravitational potential energy, so the final potential energy is $U_f = 0$. We also have no work being done by non-conservative forces. This may seem somewhat counter-intuitive at first, because static friction acts on each object as it rolls down the hill, but it is kinetic friction that is associated with a loss of mechanical energy. Static friction, because it involves no relative motion (and therefore no displacement to use in the work equation), does not produce a loss of mechanical energy.

The conservation of energy equation can thus be written: $U_i = K_f$.

Let's say that each object starts from a height h above the bottom of the incline. Because the zero for potential energy is at the bottom, the initial gravitational potential energy can be written as: $U_i = Mgh$. Our energy conservation term can thus be written $Mgh = K_f$.

Step 3 – ***Split the kinetic energy term into two pieces, one representing the translational kinetic energy and one representing the rotational kinetic energy. Express the rotational kinetic energy in terms of M and v_f (the speed at the bottom of the incline) and solve for v_f.*** First, let's think about why considering two types of kinetic energy is appropriate. When an object's center-of-mass is moving, the object has translational kinetic energy $KE_{trans} = \frac{1}{2}Mv^2$. When an object is only rotating, it has a rotational kinetic energy $KE_{rot} = \frac{1}{2}I\omega^2$. A rolling object, however, is both translating as well as rotating, and thus it has both these forms of kinetic energy.
Our energy equation now becomes: $Mgh = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega_f^2$.

Let's make two substitutions to rewrite the rotational kinetic energy term. First, we can use our expression for rotational inertia, $I = cMR^2$. Then, we use the relationship between speed and angular speed that applies to rolling without slipping,: $\omega = v/R$. Our energy equation is now:

$$Mgh = \frac{1}{2}Mv_f^2 + \frac{1}{2}cMR^2\frac{v_f^2}{R^2}.$$

Note that all factors of mass M and radius R cancel, leaving: $gh = \frac{1}{2}v_f^2 + \frac{1}{2}cv_f^2$.

Solving for v_f, the object's speed at the bottom of the incline, gives: $v_f = \sqrt{\frac{2gh}{1+c}}$.

This result is consistent with the $v_f = \sqrt{2gh}$ result we obtained in previous chapters (for the speed of a ball dropped from rest through a height h, for instance), giving us some confidence that the answer is correct.

So, which object wins the race? The winner is the object with the highest speed at the bottom, which requires the smallest value of c. Recall that c is the numerical factor in the moment of inertia, $I = cMR^2$. For the various shapes we were racing we have $c = 2/5$ for solid spheres; $c = 1/2$ for uniform disks and cylinders; and $c = 1$ for rings. Thus, in the rolling races, a solid sphere beats any disk (or cylinder) and any ring, while any disk or cylinder beats any ring.

Key ideas: We can apply energy conservation in an analysis of rotating, or rolling, objects, just as we did in previous situations. Our energy conservation equation from Chapter 7 needs no modification. All we have to do is to use the expression for the kinetic energy of rotating objects: $KE_{rot} = \frac{1}{2}I\omega^2$. **Related End-of-Chapter Exercises: 7, 8, 10.**

Essential Question 11.8: In Exploration 11.8, we determined that, in the races of rolling objects, a solid sphere would beat a disk or cylinder, which would beat a ring. What if we raced two of the same kind of object against one another (such as a sphere versus a sphere)? Which object would win? The object with the larger mass, smaller mass, larger radius, or smaller radius?