Answer to Essential Question 11.4: Once again, it is simplest to take torques about an axis perpendicular to the page, passing through the point at which the spool touches the ground. The only force giving rise to a torque about this point is the tension in the ribbon, which gives a clockwise torque. If the spool rotates clockwise with respect to its bottom point, the motion of the spool is to the right.

11-5 Analyzing the Motion of a Spool

EXPLORATION 11.5 – Continuing the analysis of the rolling spool
     Let's return to the situation described in Exploration 11.4, and focus in particular on the free-body diagram in Figure 11.12. Our goal is to determine the magnitude of the spool's acceleration in terms of F_T and M. The spool consists of two disks, each of mass $M/3$ and radius R, connected by an axle of mass $M/3$ and radius $R/2$.
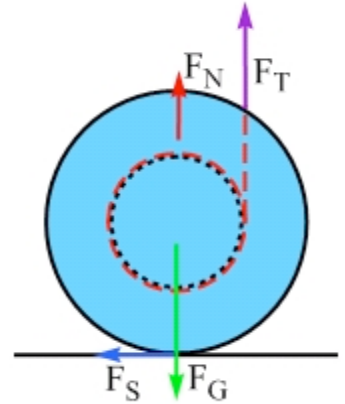
Step 1 – Apply Newton's Second Law for the horizontal forces.
     The spool accelerates left, so let's define left to be the positive direction.

$$\sum \vec{F}_x = M\,\vec{a}_x .$$

> **Figure 11.12**: The complete free-body diagram of the spool.

     Because the acceleration is entirely in the x-direction, we can replace $\vec{a}_x$ by $\vec{a}$.

Evaluating the left-hand side of this expression with the aid of Figure 11.12 gives:

$$+F_s = +M\,a .$$

Step 2 – Find the expression for the spool's rotational inertia about an axis perpendicular to the page passing through the center of the spool.
     Why are we doing this? Well, we'll need to apply Newton's second law for rotation to solve this problem, and that involves the spool's rotational inertia. To find the spool's rotational inertia, we can use the expression for the rotational inertia of a solid disk or cylinder ($I = \frac{1}{2}\,mr^2$) about the center. Let's apply this equation to the three pieces of the spool and add them together to find the net rotational inertia.

     Each of the two disks contributes $\dfrac{1}{2}\left(\dfrac{M}{3}\right)R^2 = \dfrac{1}{6}MR^2$ to the rotational inertia.

     The axle contributes $\dfrac{1}{2}\left(\dfrac{M}{3}\right)\left(\dfrac{R}{2}\right)^2 = \dfrac{1}{24}MR^2 .$

     The total rotational inertia is $I = \dfrac{1}{6}MR^2 + \dfrac{1}{6}MR^2 + \dfrac{1}{24}MR^2 = \dfrac{9}{24}MR^2 = \dfrac{3}{8}MR^2 .$

Step 3 – Apply Newton's Second Law for Rotation to obtain a connection between the force of friction and the upward force $\vec{F}_T$ applied to the string.
     Taking torques about an axis perpendicular to the page and passing through the center of the spool is a good way to do this, because the force of gravity and the normal force pass through this axis and therefore give no torque about that axis. Since the spool has a counterclockwise angular acceleration let's take counterclockwise to be positive for torques. Applying Newton's second law for rotation gives:

$$\sum \vec{\tau} = I\vec{\alpha} .$$

Referring to Figure 11.12, and using the equation $\tau = r\,F\sin\theta$, we have:

$$+\frac{R}{2}F_T\sin\left(90°\right) - R\,F_S\sin\left(90°\right) = +I\alpha\ .$$

Recognizing that $\sin\left(90°\right) = 1$, and substituting the expression for the spool's rotational inertia we found above, gives:

$$+\frac{R}{2}F_T - R\,F_S = +\frac{3}{8}MR^2\alpha\ .$$

Canceling a factor of R gives: $+\dfrac{1}{2}F_T - F_S = +\dfrac{3}{8}MR\alpha$.

Step 4 – *What is the connection between the spool's acceleration and its angular acceleration?*
For rolling without slipping, the connection between the acceleration and the angular acceleration is $a = R\alpha$, although it is always a good idea to check whether the positive direction for the straight-line motion is consistent with the positive direction for rotation. In our case they are consistent, since we chose them based on the motion. If we had reversed one of the positive directions, however, we would have had a negative sign in the equation.

Step 5 – *Combine the results above to determine the spool's acceleration in terms of F_T and M.*
Let's first substitute $a = R\alpha$ into our final expression from step 3, to get:

$$+\frac{1}{2}F_T - F_S = +\frac{3}{8}Ma\ .$$

In step 1, we determined that $F_s = M\,a$, so we get:

$$+\frac{1}{2}F_T - Ma = +\frac{3}{8}Ma\ ;$$

$$+\frac{1}{2}F_T = +\frac{11}{8}Ma\ ;$$

$$\vec{a} = \frac{4F_T}{11M}\ \text{,directed to the left.}$$

Key idea: Solving a rolling-without-slipping problem often involves analyzing the rotational motion, analyzing the one-dimensional motion, and combining the analyses.
Related End-of-Chapter Exercises: 51, 53.

Essential Question 11.5: Consider a hard ball that is rolling without slipping across a smooth level surface. If the ball maintains a constant velocity, in what direction is the static force of friction acting on the ball? Consider the three-possible free-body diagrams for the ball in Figure 11.14 below, and state which free-body diagram is appropriate for this situation.

Figure 11.14: Possible free-body diagrams for a ball rolling, without slipping, at constant velocity to the right across a horizontal surface.