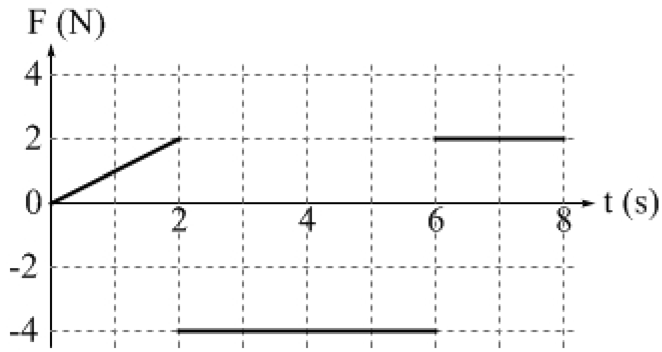
PROBLEM 1 – 10 points

At t = 0, an object of mass 0.50 kg is passing through the origin with a velocity of 8.0 m/s in the positive x direction. It is then subjected to a net force in the x-direction that varies in magnitude and direction as shown by the graph.



When the force is positive it means the force is directed in the +x direction. When the force is negative it is directed in the –x direction.

[4 points] (a) What is the object's velocity at t = 8 s?

The area under the graph represents the impulse, which is the change in momentum experienced by the object. The three different regions have areas of +2.0 kg m/s, –16.0 kg m/s, and +4.0 kg m/s, respectively. Adding them up gives a net change in momentum of $\Delta\vec{p} = -10.0 \text{ kg m/s}$ **over the entire 8-second interval.**

We also know that the initial momentum is $\vec{p}_i = m\vec{v}_i = (0.50 \text{ kg})(+8.0 \text{ m/s}) = +4.0 \text{ kg m/s}$.

The momentum at *t* = 8 s is thus $\vec{p}_f = \vec{p}_i + \Delta\vec{p} = +4.0 \text{ kg m/s} - 10.0 \text{ kg m/s} = -6.0 \text{ kg m/s}$.

The velocity at *t* = 8 s is thus $\vec{v}_f = \frac{\vec{p}_f}{m} = \frac{-6.0 \text{ kg m/s}}{0.50 \text{ kg}} = -12 \text{ m/s}$.

[3 points] (b) When does the object reach its maximum speed? Justify your answer.

The four possibilities to consider are t = 0, t = 2 s, t = 6 s, and t = 8 s. Let's find the momentum at each of these times.
At t = 0 we have $\vec{p}_i = m\vec{v}_i = (0.50 \text{ kg})(+8.0 \text{ m/s}) = +4.0 \text{ kg m/s}$.
At t = 2 s the momentum is $\vec{p}(t = 2\text{s}) = \vec{p}_i + 2.0 \text{ kg m/s} = +6.0 \text{ kg m/s}$.
At t = 6 s the momentum is $\vec{p}(t = 6\text{s}) = \vec{p}(t = 2\text{s}) - 16.0 \text{ kg m/s} = -10.0 \text{ kg m/s}$.
At t = 8 s we found in part (a) that $\vec{p}_f = -6.0 \text{ kg m/s}$.

We are just asked to find the maximum speed, so that happens at t = 6 s when the magnitude of the momentum is largest.

[3 points] (c) What is the maximum speed reached by the object?

The maximum speed is the magnitude of the maximum momentum divided by the mass:
$v_{\max} = \frac{p(t = 6\text{s})}{m} = \frac{10 \text{ kg m/s}}{0.5 \text{ kg}} = 20 \text{ m/s}$.

PROBLEM 2 – 10 points

A 60 kg man and his 40 kg dog are sitting together at the left end of a boat that is 8.0 m long. The boat's mass is 100 kg, and we can assume the boat's center of mass is in the center of the boat. The boat starts out at rest in the middle of a calm lake. Ignore all friction and water resistance throughout this problem.

[3 points] (a) Where is the center of mass of the boat + man + dog system? Indicate your answer by stating the distance from the man to the center of mass of the system.

Let's measure all distances from the man, defining right to be positive. Then, the location of both the man and the dog is 0 m, while the center of the boat is at +4 m. Hence, the center-of-mass of the system is at

$$X_{CM} = \frac{(60\text{ kg})(0\text{ m}) + (100\text{ kg})(4\text{ m}) + (40\text{ kg})(0\text{ m})}{60\text{ kg} + 100\text{ kg} + 40\text{ kg}} = \frac{400\text{ kg m}}{200\text{ kg}} = 2\text{ m}$$

[4 points] (b) Suppose that the dog moves to the other end of the boat, while the man stays still. How far does the boat move as a result? Indicate the magnitude as well as the direction (to the left or to the right) of the boat's displacement.

We'll find the new distance between the man and the center-of-mass of the system after the dog has moved. Because there are no external forces acting on the system, the center-of-mass of the system must remain at rest, and any difference in distance from the man to the system compared to part (a) must come from the fact that the boat has moved.

The dog is now 8 m away from the man. This gives, for the center of mass

$$X_{CM} = \frac{(60\text{ kg})(0\text{ m}) + (100\text{ kg})(4\text{ m}) + (40\text{ kg})(8\text{ m})}{60\text{ kg} + 100\text{ kg} + 40\text{ kg}} = \frac{720\text{ kg m}}{200\text{ kg}} = 3.6\text{ m},$$

as measured from the man. This means that the boat has moved 1.6m to the left, because the center-of-mass of the system must remain fixed, relative to a fixed point.

[3 points] (c) After reaching the other end of the boat, the dog jumps off with a horizontal velocity of 2.0 m/s, directed to the right, as measured relative to the water. What is the speed with which the boat and the man drift in the opposite direction?

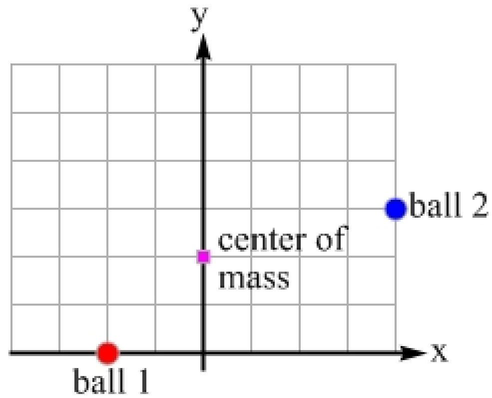
Because there are no external forces acting on the system, the velocity of the center-of-mass must remain constant. The initial velocity of the center-of-mass is zero, because nothing was moving. The final velocity of the center-of-mass is

$$V_{CM} = \frac{m_{dog}v_{dog} + M_{man+boat}V_{man+boat}}{M_{total}} = \frac{(40\text{ kg})(2\text{ m/s}) + M_{man+boat}V_{man+boat}}{M_{total}}$$

which we set equal to the initial value, zero, because momentum is conserved in this case. This gives us

$$V_{man+boat} = -\frac{(40\text{ kg})(2\text{ m/s})}{60\text{ kg} + 100\text{ kg}} = -0.5\text{ m/s}$$

The minus sign tells us that the boat and the man drift in the opposite direction from the dog. Their speed is 0.5 m/s.



PROBLEM 3 – 15 points

A system consists of three balls at different locations near the origin, as shown at right. Ball 1 has a mass of 2.0 kg and is located on the x-axis at x_1 = –2.0 m; ball 2 has an unknown mass and is located at (x_2 = +4.0 m, y_2 = +3.0 m); ball 3 is somewhere on the y-axis at an unknown location, and it has a mass of 1.0 kg.

The coordinates of the center-of-mass of this system are (x_{CM} = 0, y_{CM} = +2.0 m). The squares on the grid measure 1.0 m × 1.0 m.

[4 points] (a) Calculate the mass of ball 2.

Here, we can use the *x*-coordinate of the center-of-mass, because we know the *x*-coordinate of ball 3 is *x* = 0.

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = 0$$

For this equation to be true, the numerator must be zero. Setting the numerator to zero gives:

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = 0$$

Solving for m_2 gives:

$$m_2 = \frac{-m_1 x_1 - m_3 x_3}{x_2} = \frac{-(2.0\text{ kg}) \times (-2.0\text{ m}) - 0}{+4.0\text{ m}} = 1.0\text{ kg}$$

[4 points] (b) Find the location of ball 3.

Remember that we know the *x*-coordinate of ball 3 is *x* = 0, so we just have to solve for the *y*-coordinate. We can do this using the equation for the *y*-coordinate of the center-of-mass, which we know has a value of +2.0 m.

$$Y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

Solving for y_3 gives:

$$y_3 = \frac{(m_1 + m_2 + m_3)Y_{cm} - m_1 y_1 - m_2 y_2}{m_3} = \frac{(4.0\text{ kg}) \times (+2.0\text{ m}) - (2.0\text{ kg}) \times (0) - (1.0\text{ kg}) \times (+3.0\text{ m})}{1.0\text{ kg}} = +5.0\text{ m}$$

[3 points] (c) Let's say you can set the mass of ball 3 to be any non-negative value. By adjusting the mass of ball 3, what range of positions can the center-of-mass of this system occupy?

This gives us plenty of freedom. If we set the mass of ball 3 to infinity, for instance, then the center-of-mass of the system will move to the location of ball 3, at $x = 0$, $y = +5$ m. At the other extreme, we can reduce the mass of ball 3 to zero, in which case the center-of-mass of the system is the center-of-mass of ball 1 and ball 2, which is at $x = 0$, $y = +1$ m.

Adjusting the mass of ball 3 will move the center-of-mass between the two extremes we found above. Thus, by adjusting the mass of ball 3, the center-of-mass of the system can be anywhere on the straight line between the points (0, +1 m) and (0, +5 m). Interestingly, if we can only adjust the mass of ball 3, the center-of-mass of the system must lie on the *y*-axis.

[4 points] (d) Return to the original situation, with the center-of-mass of the system at ($x_{CM} = 0$, $y_{CM} = +2.0$ m). You now adjust the mass of one of the balls, and you observe that this adjustment causes the center of mass of the system to shift a little to the right and a little higher than $y = +2.0$ m. Which of the following changes could have caused this? **Select all that apply**.

[ ] the mass of ball 1 was increased

[**X**] the mass of ball 1 was decreased

[**X**] the mass of ball 2 was increased

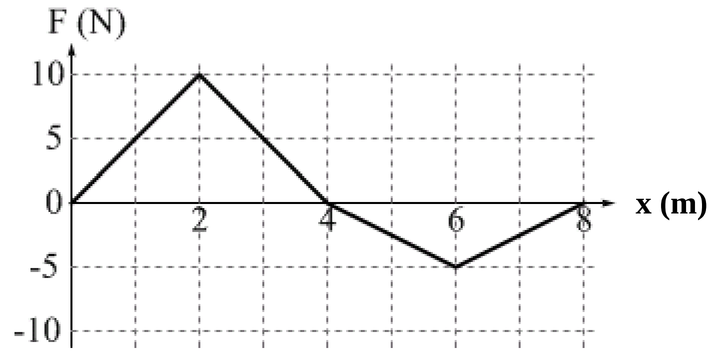
[ ] the mass of ball 2 was decreased

[ ] the mass of ball 3 was increased

[ ] the mass of ball 3 was decreased

As we discussed in part (c), changing the mass of ball 3 can change only the y-coordinate and not the x-coordinate. In this case, we're changing both coordinates, so we cannot be changing the mass of ball 3. The motion of the center-of-mass is consistent with motion away from ball 1, and motion toward ball 2. The center-of-mass will move away from ball 1 if we decrease the mass of ball 1, so that is one possibility. The center-of-mass will move toward ball 2 if we increase the mass of ball 2, so that is the other possibility.

PROBLEM 4 – 20 points



When an object with a mass of 2.0 kg passes through the origin, its velocity is 4.0 m/s in the positive *x*-direction. It is then subjected to a net force in the *x*-direction that varies in magnitude and direction as shown by the graph. Note that the graph gives force as a function of position, not force as a function of time.

When the force is positive it means the force is directed in the +*x* direction. When the force is negative it is directed in the –*x* direction.

[4 points] (a) At what location does the object reach its maximum speed? Justify your answer.

The three possibilities to consider are *x* = 0, *x* = +4 m, and *x* = +8 m. Remember that the work done on the object by the force is the area under the curve on the graph above. As the object moves toward *x* = +4 m, the net work done is more and more positive, so the object's kinetic energy, and speed, continue to increase as the object moves from *x* = 0 to *x* = +4 m. After *x* = +4 m, the work is negative, so the object slows down. The net negative work between *x* = +4 m and *x* = +8 m is smaller in magnitude than the net positive work on the object between *x* = 0 to *x* = +4 m, so the object is going fastest at *x* = +4 m, and it is moving faster at *x* = +8 m than it is at *x* = 0.

Another approach is to consider the direction of the force. Between *x* = 0 and *x* = +4 m, the force is in the same direction as the velocity, so the object speeds up. After *x* = +4 m, the force and velocity are in opposite directions, however, so the object slows down, but it never reverses direction. Thus, the object has the largest speed at *x* = +4 m.

[6 points] (b) What is the maximum speed reached by the object?

Here, we can use the fact that the area under the curve is the work done, and the work done is the change in kinetic energy.

The object's kinetic energy at x = 0 is $KE_i = \frac{1}{2}mv_i^2 = \frac{1}{2}(2.0 \text{ kg}) \times (4.0 \text{ m/s})^2 = 16 \text{ J}$ **.**

The work done on the object as it moves from *x* = 0 to *x* = +4 m is the area under the curve for that region, which we can calculate using the equation for the area of a triangle, one-half of the base multiplied by the height.

$W_{0 \to 4} = \frac{1}{2}(4.0 \text{ m}) \times (10 \text{ N}) = +20 \text{ J}$ **.**

The object's kinetic energy when it arrives at *x* = +4 m is thus:

$KE_f = KE_i + W_{0 \to 4} = 16 \text{ J} + 20 \text{ J} = 36 \text{ J}$ **.**

To solve for the object's speed at *x* = +4 m, we can use:

$$KE_f = \frac{1}{2}mv_f^2 \qquad so \qquad v_f = \sqrt{\frac{2KE_f}{m}} = \sqrt{\frac{2 \times 36 \text{ J}}{2.0 \text{ kg}}} = 6.0 \text{ m/s}$$

[5 points] (c) What is the object's kinetic energy when it reaches $x = +8$ m?

We can use a similar method here, going all the way from x = 0 to x = +8 m, or using our previous result and adding the work for the region from x = +4 m to x = +8 m. If we try the latter approach, we first find the work for the region of interest:

$W_{4\to8} = \frac{1}{2}(4.0 \text{ m})\times(-5 \text{ N}) = -10 \text{ J}$.

The object's kinetic energy when it arrives at *x* = +8 m is thus:

$KE_{x=+8} = KE_{x=+4} + W_{4\to8} = 36 \text{ J} - 10 \text{ J} = 26 \text{ J}$.

[5 points] (d) Now suppose the object started at $x = +6$ m instead of at $x = 0$. What minimum kinetic energy would the object need to have at $x = +6$ m to be able to make it to $x = 0$, assuming that its initial velocity was in the negative *x*-direction?

Traveling to the left, the net work done on the object between *x* = +6 m and *x* = +4 m is +5 J, and then from *x* = +4 m to *x* = 0 m, the net work done is –20 J. These signs come from comparing the directions of the force and the displacement. Between *x* = +6 m and *x* = +4 m, both the displacement and the force are in the negative x-direction – because the force and displacement are in the same direction, the work is positive. Between *x* = +4 m and *x* = 0 m, we get a negative sign for the work because the force (positive) and displacement (negative) are in opposite directions.

The total net work done, then, is +5 J – 20 J = –15 J. Thus, the minimum kinetic energy the object would need at *x* = +6 m is +15 J, so it would just arrive at *x* = 0, arriving there with no kinetic energy, the minimum possible.