Answer to Essential Question 5.6: As the speed increases the angle θ increases. One way to see this is to consider what happens to the tension. The vertical component of the tension remains the same, balancing the force of gravity, while the horizontal component of the tension increases (increasing the angle), because it provides the force toward the center needed for the ball to go in a circle, and that force increases as v increases. We can also convince ourselves that the angle increases, as the speed increases, by looking at the final equation in part (b) of Example 5.6.

5-7 Using Whole Vectors

The standard method of solving a problem involving Newton's laws is to break the forces into components. However, using whole vectors is an alternate approach. Let's see how whole vectors can be applied in a particular situation.

EXAMPLE 5.7 – Using whole vectors

(a) A box is placed on a frictionless ramp inclined at an angle θ with the horizontal. The box is then released from rest. Find an expression for the normal force acting on the box in this situation. What is the role of the normal force? What is the acceleration of the box?

(b) A box truck is traveling in a horizontal circle around a banked curve that is inclined at an angle θ with the horizontal. The curve is covered with ice and is effectively frictionless, so the truck can make it safely around the curve only if it travels at a particular constant speed (known as the *design speed* of the curve). Find an expression for the normal force acting on the truck in this situation. What is the role of the normal force? What is the design speed of the curve?

(c) Compare and contrast these two situations.

SOLUTION

(a) As usual, our first step is to draw a diagram, and then a free-body diagram showing the forces acting on the box, as in Figure 5.18. The two forces are the downward force of gravity, and the normal force applied by the ramp to the box. Because we are using whole vectors, we don't need to worry about splitting vectors into components. It is crucial, however, to think about the direction of the acceleration, which in this case is directed down the slope.

Let's apply Newton's second law, $\sum \vec{F} = m\vec{a}$, adding the forces as vectors. In this case, we get the right-angled triangle in Figure 5.18c. Each side of the triangle represents one vector in the equation $m\vec{g} + \vec{F}_N = m\vec{a}$. The vector $m\vec{a}$ (the net force) is parallel to the ramp, while the normal force is perpendicular to the ramp and the force of gravity is directed straight down. θ, the angle of the ramp, is the angle at the bottom of the triangle.

Because the force of gravity is on the hypotenuse of the triangle, we get:

$$\cos\theta = \frac{F_N}{mg}, \qquad \text{so} \qquad F_N = mg\cos\theta .$$

The role of the normal force is simply to prevent the box from falling through the ramp.

We can find the acceleration from the geometry of the right-angled triangle:

$$\sin\theta = \frac{ma}{mg} = \frac{a}{g} .$$

This relationship gives $\vec{a} = g\sin\theta$ directed down the slope.

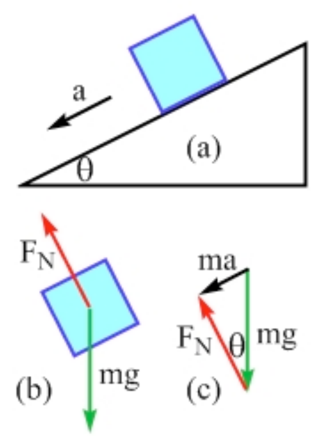

Figure 5.18: (a) A diagram, (b) free-body diagram, and (c) a right-angled triangle to show how the force of gravity and normal force combine to give the net force on the box.

(b) The situation of the box truck traveling around the banked curve resembles the box on the incline. A diagram of the situation, showing the back of the truck, is illustrated in Figure 5.19. The same forces, the force of gravity and the normal force, appear on this free-body diagram as in the free-body diagram for the box. The difference lies in the direction of the acceleration, which for the circular motion situation is directed horizontally to the left, toward the center of the circle.

Again we apply Newton's second law, $\sum \vec{F} = m\vec{a}$, adding the forces as vectors, where the magnitude of the acceleration has the special form $a_c = v^2 / r$. This gives:

$m\vec{g} + \vec{F}_N = \dfrac{mv^2}{r}$, directed toward the center of the circle.

Again, each force represents one side of a right-angled triangle. Now the normal force is on the hypotenuse, and clearly must be larger than it is in part (a).

Using what we know about the geometry of right-angled triangles, we get:

$$\cos\theta = \frac{mg}{F_N}, \qquad \text{so now } F_N = \frac{mg}{\cos\theta}.$$

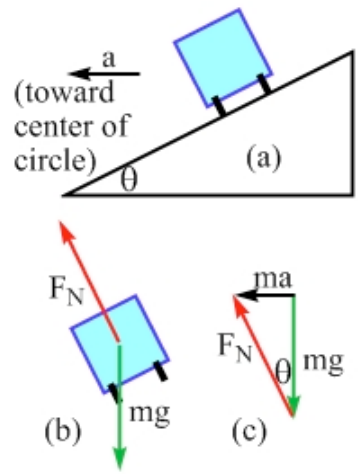

Figure 5.19: (a) A diagram, (b) free-body diagram, and (c) right-angled triangle to show how the force of gravity and normal force combine to give the net force on the truck.

If the angle of the ramp is larger than zero, then $\cos\theta$ is a number less than 1 and $F_N > mg$. In part (a) we had $F_N < mg$. The normal force for the box truck is larger because it has two roles. Not only does the normal force prevent the truck from falling through the incline, it must also provide the force directed toward the center to keep the truck moving around the circle.

To find the design speed of the curve, we can use the other side of the triangle:

$$\sin\theta = \frac{ma}{F_N} = \frac{mv^2}{rF_N} = \frac{mv^2\cos\theta}{rmg} = \frac{v^2\cos\theta}{rg}.$$

Re-arranging the preceding equation gives a design speed of $v = \sqrt{\dfrac{rg\sin\theta}{\cos\theta}} = \sqrt{rg\tan\theta}$.

This is an interesting result. First, there is a design speed, a safest speed to negotiate the curve. At the design speed, the vehicle needs no assistance from friction to travel around the circle. Going significantly faster is dangerous because the vehicle is only prevented from sliding toward the outside of the curve by the presence of friction - the faster you go, the larger the force of friction required. Second, the design speed does not depend on the vehicle mass, which is fortunate for the road designers. The same physics applies to a Mini Cooper as to a large truck.

(c) A key similarity is the free-body diagram: in both cases there is a downward force of gravity and a normal force perpendicular to the slope. The key difference is that the accelerations are in different directions, requiring a larger normal force in the circular motion situation.

Related End-of-Chapter Exercises: 25 – 27.

Essential Question 5.7: Consider again the situation of the truck on the banked curve. In icy conditions, is it safest to drive very slowly around the curve or to drive at the design speed?